EXAMPLE GRAMMAR SCHOOL

ESAT Mock: Worked Solutions

Mathematics 2

Paper SMP-2026-09 · Version 1.0

Staff copy

This document contains the answers. Keep it away from candidates until the paper has been sat. Marking guidance is in the separate mark scheme.

Question 1

MM1.1 · Algebra · ★★☆☆☆ · 30 s

Simplify $\dfrac{x^{5/2} \cdot x^{-1/2}}{x^{3/2}}$.

- **A.** 1
- **B.** x^2
- **C.** $x^{3/2}$
- **D.** x
- **E.** $x^{1/2}$
- **F.** $x^{7/2}$
- **G.** $x^{9/2}$
- **H.** $x^{-1/2}$

Answer E. Key idea

Numerator: $x^{5/2+(-1/2)} = x^2$. Then $x^2/x^{3/2} = x^{1/2}$.

Worked solution

1. Simplify the numerator

$$x^{5/2} \cdot x^{-1/2} = x^{5/2-1/2} = x^{4/2} = x^2$$

2. Divide by the denominator

$$\frac{x^2}{x^{3/2}} = x^{2-3/2} = x^{1/2}$$

Matches Option **E**.

Why the other options are wrong

- **A** (1): Arises from a common inversion, algebraic sign error, or omitting an intermediate conversion factor. *(Calculation / Conceptual Error)*
- **B** (x^2): Arises from a common inversion, algebraic sign error, or omitting an intermediate conversion factor. *(Calculation / Conceptual Error)*
- **C** ($x^{3/2}$): Arises from a common inversion, algebraic sign error, or omitting an intermediate conversion factor. *(Calculation / Conceptual Error)*

- **D** (x): The $x^{-1/2}$ factor is lost when the fraction is copied, leaving $x^{5/2}/x^{3/2} = x^{5/2-3/2} = x^1$, printing x. The missing factor is worth $-1/2$ on the index, which is exactly the difference between x and the key $x^{1/2}$. *(Factor dropped from the numerator)*
- **F** ($x^{7/2}$): Added the denominator exponent instead of subtracting it. *(Sign Error)*
- **G** ($x^{9/2}$): Every index is added and both negatives are dropped, the $-1/2$ in the numerator and the subtraction owed to the denominator: $5/2 + 1/2 + 3/2 = 9/2$, printing $x^{9/2}$. This is one step past the $x^{7/2}$ slip, which at least keeps the sign on the $-1/2$. *(Both minus signs ignored)*
- **H** ($x^{-1/2}$): The numerator is simplified correctly to $x^{5/2-1/2} = x^2$, then the last subtraction is taken the wrong way round: the numerator index is subtracted from the denominator index, $3/2 - 2 = -1/2$, printing $x^{-1/2}$. Dividing takes numerator index minus denominator index, $2 - 3/2 = 1/2$. *(Subtraction reversed at the division step)*

Common mistake. Adding all the exponents (including the denominator): $5/2 - 1/2 + 3/2 = 7/2$.

Takeaway. **The Cheat Code:** Always inspect units and algebraic structure before calculating. Simplifying fractions and factoring terms early reduces computation time by more than 50%.

Time-saving tip. Speed Heuristic: Look at the relative magnitudes and units in the multiple-choice options to eliminate extreme or impossible values immediately.

Question 2

MM8.3 · Graphs and transformations · ★★★☆☆ · 90 s

A pressure sensor's calibration line is $y = 2x + 5$. After the sensor is re-tuned its calibration line has a gradient 50% larger and an intercept on the y-axis 3 units lower. Give the coordinates of the point where the two calibration lines meet.

A. $(2, 9)$
B. $(-3, -1)$
C. $(-7, -9)$
D. $(5, 15)$
E. $(-3, -4)$
F. $(3, 11)$
G. $(0, 5)$
H. $(6, 17)$

Answer F. Key idea

Altering m and altering c do two different things to a straight line. The gradient m controls steepness, so changing it pivots the line about the point where it crosses the y-axis, while c is that crossing itself, so changing it slides every point of the line vertically by the same amount. Once both parameters of the second line are known, the point the two lines have in common is found by setting their expressions for y equal.

Worked solution

Step 1: Read the two parameters of the original line.

Written in the form $y = mx + c$, the line $y = 2x + 5$ has $m = 2$ and $c = 5$, so its gradient is 2 and it crosses the y-axis at $(0, 5)$.

Step 2: Alter m.

A gradient 50% larger means multiplying by $\frac{3}{2}$, so $m' = \frac{3}{2} \times 2 = 3$. This change pivots the line about its y-axis crossing, so the crossing is still at 5 after this step alone.

Step 3: Alter c.

Lowering the intercept by 3 gives $c' = 5 - 3 = 2$, a pure vertical translation. The re-tuned calibration line is therefore $y = 3x + 2$.

Step 4: Set the two expressions for y equal.

$2x + 5 = 3x + 2$, so $5 - 2 = 3x - 2x$, giving $x = 3$.

Step 5: Find y and check on both lines.

The original line gives $y = 2(3) + 5 = 11$ and the re-tuned line gives $y = 3(3) + 2 = 11$, so the meeting point is $(3, 11)$. Sanity check: at $x = 0$ the new line sits 3 below the old one, and it gains $3 - 2 = 1$ unit on it for every unit of x, so it needs exactly 3 units of x to catch up, which agrees with $x = 3$.

Matches Option **F**.

Why the other options are wrong

- **A** ($(2,\ 9)$): Reading '3 units lower' as an intercept of 3 rather than $5 - 3 = 2$ gives $y = 3x + 3$. Then $2x + 5 = 3x + 3$ gives $x = 2$, and $y = 2(2) + 5 = 9$, printing $(2,\ 9)$. *(Misread of the intercept change)*
- **B** ($(-3,\ -1)$): Lifting the intercept instead of lowering it gives $c' = 5 + 3 = 8$ and the line $y = 3x + 8$. Then $2x + 5 = 3x + 8$ gives $-3 = x$, and $y = 2(-3) + 5 = -1$, printing $(-3,\ -1)$. *(Wrong direction for the change in c)*
- **C** ($(-7,\ -9)$): A sign slip when collecting terms: writing $2x - 3x = 2 + 5$ instead of $2x - 3x = 2 - 5$ gives $-x = 7$, so $x = -7$ and $y = 2(-7) + 5 = -9$, printing $(-7,\ -9)$. *(Sign error moving a constant across the equals sign)*
- **D** ($(5,\ 15)$): Steepening is taken to swing the line about $(0, 0)$ rather than about its y-axis crossing, so the re-tuned line is written as $y = 3x$ with no intercept at all. Then $2x + 5 = 3x$ gives $x = 5$, and $y = 2(5) + 5 = 15$, printing $(5,\ 15)$. Changing m pivots about $(0, 5)$, and the intercept only moves when c is altered. *(Line pivoted about the origin)*
- **E** ($(-3,\ -4)$): The intercept drop is applied to the original line and only the steepening to the re-tuned one, giving $y = 2x + 2$ and $y = 3x + 5$. Then $2x + 2 = 3x + 5$ gives $x = -3$, and $y = 2(-3) + 2 = -4$, printing $(-3,\ -4)$. Both changes describe the line after re-tuning, so both belong to the same equation. *(The two changes applied to opposite lines)*
- **G** ($(0,\ 5)$): Applying only the gradient change leaves c at 5, giving $y = 3x + 5$. Two lines sharing a y-intercept meet on the y-axis, so $2x + 5 = 3x + 5$ gives $x = 0$ and $y = 5$, printing $(0,\ 5)$. *(Second alteration omitted)*
- **H** ($(6,\ 17)$): The gradient 50% larger is read as adding 0.5 to it rather than multiplying by $\frac{3}{2}$, giving $m' = 2 + 0.5 = 2.5$ and the line $y = 2.5x + 2$. Then $2x + 5 = 2.5x + 2$ gives $0.5x = 3$, so $x = 6$ and $y = 2(6) + 5 = 17$, printing $(6,\ 17)$. Fifty per cent of 2 is 1, so the new gradient is 3. *(Percentage added instead of multiplied)*

Common mistake. Assuming that steepening a line also moves where it crosses the y-axis. Changing m pivots the line about $(0, c)$, so the crossing stays at 5 until c itself is altered. Candidates who instead pivot about the origin write the new line as $y = 3x$, solve $2x + 5 = 3x$ and report $(5, 15)$.

Takeaway. The Cheat Code: In $y = mx + c$, m is a pivot and c is a lift: change m and the line turns about $(0, c)$, change c and every point slides vertically. Two lines then close their intercept gap at a rate equal to the difference of their gradients.

Time-saving tip. Speed Heuristic: Skip the simultaneous equations. The gradients differ by 1 and the intercepts differ by 3, so the lower line closes the gap at 1 per unit of x and they meet at $x = 3$; one substitution gives $y = 11$.

Question 3

MM6.2 · Differentiation · ★★★☆☆ · 90 s

A lens surface is machined so that its cross-section satisfies $y = \dfrac{3x^6 + 5x^2\sqrt{x} - 2}{x^2}$ for $x > 0$, with x and y in millimetres. Find the gradient of the cross-section at $x = 1$.

A. $\dfrac{49}{2}$
B. 21
C. 11
D. $\dfrac{61}{4}$
E. $\dfrac{21}{2}$
F. $\dfrac{37}{2}$
G. $\dfrac{29}{2}$
H. $\dfrac{77}{4}$

Answer F. Key idea

The power rule differentiates x^n to nx^{n-1} for any rational n, positive, negative or fractional, but it applies only to a sum of separate powers. An expression written as one fraction is not in that form, so the first move is to divide the denominator into each term and rewrite roots and reciprocals as indices. Only then does term by term differentiation apply, and a gradient at a particular point is the value of that derivative there.

Worked solution

1. Simplify into a sum of powers

Dividing each term of the numerator by x^2, and using $x^2\sqrt{x} = x^{5/2}$:

$$y = \frac{3x^6}{x^2} + \frac{5x^{5/2}}{x^2} - \frac{2}{x^2} = 3x^4 + 5x^{1/2} - 2x^{-2}$$

In plainer form this is $y = 3x^4 - \dfrac{2}{x^2} + 5\sqrt{x}$.

2. Differentiate term by term

$$\frac{dy}{dx} = 3(4)x^3 + 5\left(\tfrac{1}{2}\right)x^{-1/2} - 2(-2)x^{-3}$$

$$\frac{dy}{dx} = 12x^3 + 4x^{-3} + \frac{5}{2}x^{-1/2} = 12x^3 + \frac{4}{x^3} + \frac{5}{2\sqrt{x}}$$

Note the middle sign: a negative coefficient multiplied by a negative index gives a positive term.

3. Substitute the required value

At $x = 1$ every power of x is 1, so

$$\frac{dy}{dx} = 12(1) + \frac{4}{1} + \frac{5}{2 \times 1} = 12 + 4 + \frac{5}{2}$$

4. Combine

$$12 + 4 + \frac{5}{2} = 16 + \frac{5}{2} = \frac{32 + 5}{2} = \frac{37}{2}$$

Sanity check on the simplification: multiplying back, $x^2\left(3x^4+5x^{1/2}-2x^{-2}\right)=3x^6+5x^{5/2}-2$, which is the numerator given. All three derivative terms are positive at $x=1$, so the surface is rising steeply there, and a gradient of $\frac{37}{2}$, that is 18.5, is consistent with the dominant $12x^3$ term.

Matches Option **F**.

Why the other options are wrong

- **A** ($\frac{49}{2}$): The x^2 is cancelled against $3x^6$ and the rest of the numerator is left as it stands, giving $y=3x^4+5x^{5/2}-2$. Differentiating that gives $12x^3+\frac{25}{2}x^{3/2}$ with the constant vanishing, and at $x=1$ the value is $12+\frac{25}{2}=\frac{49}{2}$. Every term of the numerator must be divided by x^2. *(Denominator divided into the first term only)*
- **B** (21): Writes the derivative of $5x^{1/2}$ as $5x^{-1/2}$, lowering the index without multiplying by it, so the last term contributes 5 instead of $\frac{5}{2}$ and the total is $12+4+5=21$. The rule multiplies by the old index as well as reducing it. *(Index reduced but coefficient not multiplied)*
- **C** (11): $x^2\sqrt{x}$ is read as $x^{2\times1/2}=x$ instead of $x^{2+1/2}=x^{5/2}$, so the middle term becomes $\frac{5x}{x^2}=5x^{-1}$, whose derivative $-5x^{-2}$ is worth -5 at $x=1$. The total is $12-5+4=11$. Multiplying powers of the same base adds the indices. *(Indices multiplied when powers are multiplied)*
- **D** ($\frac{61}{4}$): Differentiates top and bottom separately: the numerator gives $18x^5+\frac{25}{2}x^{3/2}$ and the denominator gives $2x$, and at $x=1$ that is $\frac{18+12.5}{2}=\frac{61}{4}$. Differentiation does not distribute over a division, so the fraction must be simplified into separate powers first. *(Quotient differentiated piecewise)*
- **E** ($\frac{21}{2}$): Differentiates $-2x^{-2}$ to $-4x^{-3}$, keeping the minus sign, so the value becomes $12-4+\frac{5}{2}=\frac{21}{2}$. Bringing the index -2 down onto the coefficient -2 multiplies two negatives, so the term is $+4x^{-3}$ and contributes $+4$. *(Sign error on the reciprocal term)*
- **G** ($\frac{29}{2}$): Reads $-\frac{2}{x^2}$ as a constant because its numerator carries no x, so the middle term is dropped and the total is $12+0+\frac{5}{2}=\frac{29}{2}$. The variable sits in the denominator, so the term is $-2x^{-2}$ and its derivative is $+4x^{-3}$, worth 4 at $x=1$. *(Constant-looking term differentiated to zero)*
- **H** ($\frac{77}{4}$): Dividing by x^2 is carried out by halving each index rather than subtracting 2: x^6 becomes x^3 and $x^{5/2}$ becomes $x^{5/4}$, giving $y=3x^3+5x^{5/4}-2x^{-2}$. Its derivative $9x^2+\frac{25}{4}x^{1/4}+4x^{-3}$ is $9+\frac{25}{4}+4=\frac{77}{4}$ at $x=1$. Division subtracts indices, so the terms are x^4 and $x^{1/2}$. *(Indices divided instead of subtracted)*

Common mistake. Differentiating the numerator and the denominator separately, as though a quotient could be handled a piece at a time. Dividing through first is what makes the power rule legal here, and it costs one line because every term of the numerator divides cleanly by x^2.

Takeaway. The Cheat Code: Get the expression into the form sum of kx^n before differentiating anything. Divide out the denominator, write roots as fractional indices and reciprocals as negative indices, and the whole problem reduces to applying nx^{n-1} three times.

Time-saving tip. Speed Heuristic: The gradient is wanted at $x=1$, where every power equals 1, so only the coefficients matter. Simplify to $3x^4+5x^{1/2}-2x^{-2}$ and read off 3×4, $5\times\frac{1}{2}$ and $(-2)\times(-2)$, then add: no powers need evaluating at all.

Question 4

MM1.2 · Algebra and functions · ★★★☆☆ · 85 s

Rationalise the denominator of $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ and simplify fully.

A. $\dfrac{5+\sqrt{21}}{2}$

B. $\dfrac{5-\sqrt{21}}{2}$

C. $\dfrac{10-\sqrt{21}}{2}$

D. $\dfrac{5-\sqrt{21}}{5}$

E. $\dfrac{2-\sqrt{21}}{2}$

Answer B. Key idea

A denominator containing a sum of two surds is cleared by multiplying above and below by its conjugate, because a conjugate pair multiplies out as a difference of two squares and leaves a rational number behind. When the numerator happens to be that same conjugate, the multiplication squares it, so the cross term of the expansion survives and the simplified answer still carries a surd on top.

Worked solution

1. Multiply by the conjugate of the denominator

The denominator is $\sqrt{7}+\sqrt{3}$, whose conjugate is $\sqrt{7}-\sqrt{3}$. Multiplying numerator and denominator by the same quantity leaves the value unchanged:

$$\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} = \frac{(\sqrt{7}-\sqrt{3})(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})} = \frac{(\sqrt{7}-\sqrt{3})^2}{(\sqrt{7})^2-(\sqrt{3})^2} = \frac{(\sqrt{7}-\sqrt{3})^2}{7-3}$$

The denominator is now the rational number 4.

2. Expand the numerator and divide through

$$(\sqrt{7}-\sqrt{3})^2 = 7-2\sqrt{7}\sqrt{3}+3 = 10-2\sqrt{21}$$

so the fraction becomes

$$\frac{10-2\sqrt{21}}{4} = \frac{2\left(5-\sqrt{21}\right)}{4} = \frac{5-\sqrt{21}}{2}$$

Both terms of the numerator carry the factor 2, so the factor cancels out of the whole bracket, not out of one term only.

Sanity check

With $\sqrt{7}\approx 2.646$ and $\sqrt{3}\approx 1.732$, the original quotient is about $0.914 \div 4.378 \approx 0.209$. Using $\sqrt{21}\approx 4.583$, the answer gives $(5-4.583)\div 2 \approx 0.209$, which agrees. The value must also be positive and smaller than 1, since the numerator is the smaller of the two surd expressions, and it is.

Matches Option **B**.

Why the other options are wrong

- **A** ($\dfrac{5+\sqrt{21}}{2}$): The cross term was expanded with the wrong sign: $(\sqrt{7}-\sqrt{3})^2$ was written as $7+2\sqrt{21}+3 = 10+2\sqrt{21}$. Dividing by the correct denominator 4 then gives $\dfrac{10+2\sqrt{21}}{4} = \dfrac{5+\sqrt{21}}{2} \approx 4.79$, which is far larger than 1 even though the original fraction has the smaller expression on top. *(Cross-term sign slip)*
- **C** ($\dfrac{10-\sqrt{21}}{2}$): From the correct $\dfrac{10-2\sqrt{21}}{4}$, the factor 2 was cancelled against the 4 but applied only to the surd term, leaving $\dfrac{10-\sqrt{21}}{2}$. The 10 must be halved as well, since the whole numerator is being divided. *(Partial cancellation)*
- **D** ($\dfrac{5-\sqrt{21}}{5}$): The denominator was multiplied out as $\sqrt{7}\times\sqrt{7}+\sqrt{3}\times\sqrt{3} = 7+3 = 10$ rather than as the difference of two squares $7-3=4$. That gives $\dfrac{10-2\sqrt{21}}{10} = \dfrac{5-\sqrt{21}}{5} \approx 0.083$, which is well below the true value of about 0.209. *(Denominator added instead of differenced)*
- **E** ($\dfrac{2-\sqrt{21}}{2}$): The second square was subtracted rather than added when expanding the numerator: $7-2\sqrt{21}-3 = 4-2\sqrt{21}$. Dividing by 4 gives $\dfrac{4-2\sqrt{21}}{4} = \dfrac{2-\sqrt{21}}{2} \approx -1.29$, a negative value, although both $\sqrt{7}-\sqrt{3}$ and $\sqrt{7}+\sqrt{3}$ are positive. *(Second square subtracted in the expansion)*

Common mistake. Multiplying numerator and denominator by the denominator itself instead of by its conjugate. That produces $\dfrac{(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})}{(\sqrt{7}+\sqrt{3})^2} = \dfrac{4}{10+2\sqrt{21}}$, which still has a surd underneath, so nothing has been rationalised and the work has to be started again.

Takeaway. The Cheat Code: When the top and bottom of a surd fraction are conjugates of one another, one multiplication does everything: the denominator collapses to a plain integer and the numerator becomes a perfect square. Expect the answer in the form $\dfrac{p-q\sqrt{r}}{s}$, and divide every term of the numerator by that integer.

Time-saving tip. Speed Heuristic: compute the denominator first. Here $(\sqrt{7})^2-(\sqrt{3})^2 = 4$, and the expanded numerator $10-2\sqrt{21}$ is even, so the printed answer must sit over 2, which discards the option written over 5 at once. A single decimal estimate near 0.21 then separates the four remaining forms without any further algebra.

Question 5

MM7.4 · Integration · ★★☆☆☆ · 90 s

Functions f and g are integrable on $1 \le x \le 9$. The integral of f is 14 over $1 \le x \le 5$ and 22 over $5 \le x \le 9$. Given $\int_1^9 (f(x)+g(x))\,dx = 92$, find $\int_1^9 g(x)\,dx$.

A. 74
B. 84
C. 56
D. 70
E. 128
F. 100
G. 78
H. 36

Answer C. Key idea

> Definite integrals over ranges that meet end to end add to give the integral over the joined range, and over one shared range the integral of a sum of functions equals the sum of the two integrals. Both facts come from areas stacking, so a question built on them needs no formula for either function.

Worked solution

1. Join the two ranges for f

$[1,5]$ and $[5,9]$ are contiguous: they meet at $x = 5$ and between them cover $[1,9]$, so their integrals add.

2. Total the integral of f

$$\int_1^9 f(x)\,dx = \int_1^5 f(x)\,dx + \int_5^9 f(x)\,dx = 14 + 22 = 36$$

3. Split the combined integrand over the equal range

Both integrals now run over the same range $[1,9]$, so

$$\int_1^9 (f(x) + g(x))\,dx = \int_1^9 f(x)\,dx + \int_1^9 g(x)\,dx$$

giving $92 = 36 + \int_1^9 g(x)\,dx$.

4. Solve for the unknown integral

$$\int_1^9 g(x)\,dx = 92 - 36 = 56$$

Sanity check

Rebuilding the combined value from the two parts: $36 + 56 = 92$, the figure given in the question.

Matches Option **C**.

Why the other options are wrong

- **A** (74): Because $[1,5]$ and $[5,9]$ each cover half of $[1,9]$, the two values are averaged rather than added: $\dfrac{14+22}{2} = 18$ is taken as $\int_1^9 f$, and $92 - 18 = 74$. Each figure is already the whole integral over its own sub-range, so ranges that meet end to end add: $14 + 22 = 36$. *(Contiguous pieces averaged instead of added)*
- **B** (84): The contiguous pieces were subtracted rather than added, $22 - 14 = 8$, and then $92 - 8 = 84$. Ranges that join end to end add. *(Contiguous ranges subtracted)*
- **D** (70): Only the second piece was removed, $92 - 22 = 70$, leaving out the 14 that f contributes over $[1,5]$. *(First sub-range dropped)*
- **E** (128): The two pieces of f are joined correctly to $14 + 22 = 36$, then $92 = 36 + \int_1^9 g$ is rearranged by adding rather than subtracting: $92 + 36 = 128$. The integral of g is what is left of the combined 92 after f's share is removed, so it must be smaller than 92, not larger. *(Sign slip rearranging for the unknown)*
- **F** (100): Written out in full the equation is $92 = 14 + 22 + \int_1^9 g$. Moving the 14 across correctly but carrying the 22 over with its sign unchanged gives $\int_1^9 g = 92 - 14 + 22 = 100$. Both pieces belong to f, so both are subtracted. *(Only one piece of f changed sign)*
- **G** (78): Only the first piece was removed, $92 - 14 = 78$, leaving out the 22 that f contributes over $[5,9]$. *(Second sub-range dropped)*
- **H** (36): The work stopped after joining the contiguous pieces: $14 + 22 = 36$ is $\int_1^9 f(x)\,dx$, and the question asks for the integral of g. *(Answered the wrong quantity)*

Common mistake. Subtracting the two pieces of f instead of adding them. The ranges $[1, 5]$ and $[5, 9]$ follow one after the other rather than one sitting inside the other, so their values combine as $14 + 22$.

Takeaway. **The Cheat Code:** Two rules cover this whole family: ranges that meet end to end add their integrals, and over one shared range the integral of a sum is the sum of the integrals. Neither rule asks what f or g actually is.

Time-saving tip. Speed Heuristic: Collapse the two pieces of f to 36 before reading anything else, and the question reduces to the single subtraction $92 - 36$.

Question 6

MM5.1 · Exponentials and logarithms · ★★★☆☆ · 90 s

A curve $y = a^x$ has $a > 1$. The chord joining its points at $x = 1$ and $x = 4$ has gradient 26. Find y when $x = -2$.

- **A.** 9
- **B.** $\frac{1}{6}$
- **C.** $\frac{1}{3}$
- **D.** $\frac{1}{81}$
- **E.** $-\frac{1}{9}$
- **F.** $\frac{1}{27}$
- **G.** -9
- **H.** $\frac{1}{9}$

Answer H. Key idea

A curve $y = a^x$ with $a > 1$ is completely determined by its base: it passes through $(0, 1)$, multiplies its height by a at every unit step to the right, and stays strictly positive because a positive base raised to any power is positive. That means the coordinates of any point on it are (x, a^x), so a geometric condition such as the gradient of a chord becomes an ordinary algebraic equation in a. Negative values of x are handled by the reciprocal rule $a^{-n} = \frac{1}{a^n}$, which is why the curve falls towards the axis on the left without ever reaching it.

Worked solution

1. Write down the two points

Every point of the curve has the form $(x,\ a^x)$, so the chord joins

$(1,\ a)$ and $(4,\ a^4)$

2. Turn the gradient into an equation in the base

$$\text{gradient} = \frac{a^4 - a}{4 - 1} = 26$$

$$a^4 - a = 78$$

3. Solve for the base

Factorise the left side as $a(a^3 - 1) = 78$. Testing simple bases greater than one: $a = 2$ gives $2(8 - 1) = 14$, which is too small, and $a = 3$ gives $3(27 - 1) = 3 \times 26 = 78$, which is exact. So $a = 3$.

This is the only solution with $a > 1$, because both a and $a^3 - 1$ increase as a increases, so their product is strictly increasing and can hit 78 only once.

4. Evaluate the curve at the required point

$y = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Sanity check: the curve is $y = 3^x$, which passes through $(1, 3)$ and $(4, 81)$, and the chord between them has gradient $\frac{81-3}{3} = \frac{78}{3} = 26$ as required. At $x = -2$ the curve lies to the left of $(0, 1)$, so the height must be positive and smaller than 1, and $\frac{1}{9}$ is.

Matches Option **H**.

Why the other options are wrong

- **A** (9): Evaluates $3^2 = 9$, ignoring the minus sign in $x = -2$. That is the height of the curve at $x = +2$, and the two are reciprocals of one another, so the required value is $\frac{1}{9}$. *(Sign of the exponent dropped)*
- **B** ($\frac{1}{6}$): Finds $a = 3$ correctly, then reads 3^{-2} as $\frac{1}{3\times 2} = \frac{1}{6}$, multiplying the base by the index instead of raising it. The index counts repeated multiplication, so $3^2 = 9$ and the height is $\frac{1}{9}$. *(Index treated as a multiplier)*
- **C** ($\frac{1}{3}$): Works back from the known point $(1, 3)$, where each unit step to the left divides the height by 3. The gap from $x = 1$ to $x = -2$ is three steps, giving $3 \div 3^3 = \frac{1}{9}$, but counting it as $|-2| = 2$ steps gives $3 \div 3^2 = \frac{1}{3}$. The number of steps is $1 - (-2) = 3$, not the size of the target x. *(Steps back along the curve miscounted)*
- **D** ($\frac{1}{81}$): Carries the value $3^4 = 81$ across from the trial in the previous step and writes $\frac{1}{81}$. That is the height at $x = -4$, not $x = -2$; the required power is $3^2 = 9$. *(Wrong power reciprocated)*
- **E** ($-\frac{1}{9}$): Finds $a = 3$ correctly, then writes $3^{-2} = -\frac{1}{3^2} = -\frac{1}{9}$, applying the reciprocal rule and carrying the minus sign into the value on top of it. The negative index does one job only, inverting the power, and $y = 3^x$ is positive at every x, so the height is $+\frac{1}{9}$. *(Reciprocal taken and the minus kept as well)*
- **F** ($\frac{1}{27}$): Finds $a = 3$, then evaluates with the run of the chord, $4 - 1 = 3$, in place of the required x, giving $3^{-3} = \frac{1}{27}$. The 3 was used up in forming the gradient equation $a^4 - a = 78$; the point asked for is $x = -2$, so the power needed is $3^2 = 9$. *(Chord run substituted for the required x)*
- **G** (-9): Interprets 3^{-2} as $-3^2 = -9$. A negative index inverts, it does not negate: $y = 3^x$ is positive for every x, so no point of the curve can lie below the horizontal axis. *(Negative index read as a negative value)*

Common mistake. Treating the chord gradient as though the curve were a straight line and writing $a = 26$, or dividing by the wrong run. The rise is the difference of two powers, $a^4 - a$, and the run is $4 - 1 = 3$, so the base satisfies a quartic condition rather than reading off directly.

Takeaway. The Cheat Code: Any point on $y = a^x$ is (x, a^x), so geometric statements about the graph become index equations in a. Once the base is known, a negative exponent is only a reciprocal: $a^{-n} = \frac{1}{a^n}$, never a negative height.

Time-saving tip. Speed Heuristic: Write $a^4 - a = 78$ as $a(a^3 - 1) = 78$ and split $78 = 3 \times 26$. Since $26 = 27 - 1 = 3^3 - 1$, the base $a = 3$ falls out with no trial at all, and the answer is then just $\frac{1}{3^2}$.

Question 7

MM1.3 · Algebra and functions · ★★★☆☆ · 90 s

A horizontal line $y = k$ never meets the curve $y = x^2 - 2x - 7$. Find all possible values of k.

- **A.** $k < -8$
- **B.** $k < -6$
- **C.** $k > -8$
- **D.** $k > 8$
- **E.** $k < -7$
- **F.** $k > -6$
- **G.** $k < -11$
- **H.** $k \leq -8$

Answer A. Key idea

A horizontal line $y = k$ crosses a curve exactly where the curve's expression equals k, so "never meets" is the statement that $x^2 - 2x - 7 = k$ has no real solution. There are two equivalent ways to pin that down. Completing the square rewrites an upward parabola as a square plus a constant, and since a square is never negative, that constant is the lowest height the curve ever reaches; heights below it are the ones the curve misses. The discriminant says the same thing formally: rearrange to the form $ax^2 + bx + c = 0$ and demand $b^2 - 4ac < 0$. Both routes must give the same inequality, and using one to check the other is the safest way to get the direction right.

Worked solution

1. Complete the square

$y = x^2 - 2x - 7$ $y = (x-1)^2 - (-1)^2 - 7$ $y = (x-1)^2 - 1 - 7$ $y = (x-1)^2 - 8$

The bracket $(x-1)^2$ is never negative and equals 0 at $x = 1$, so the lowest point of the curve is $(1, -8)$ and its least y-value is -8.

2. Turn "never meets" into a condition on the roots

The line $y = k$ meets the curve wherever $x^2 - 2x - 7 = k$, so "never meets" means that equation has no real solution. Written in standard form,

$$x^2 - 2x - (7 + k) = 0$$

so the discriminant is

$$D = (-2)^2 - 4(1)(-(7+k)) = 4 + 4(7+k) = 32 + 4k$$

No real roots requires $D < 0$:

$$32 + 4k < 0 \implies 4k < -32 \implies k < -8$$

3. Read it off the graph and check

The completed square gives the same test in one line: $(x-1)^2 = k + 8$ can be solved only when $k + 8 \geq 0$, so the line misses the curve exactly when $k < -8$.

Sanity check: $k = -9$ lies below the lowest point of the curve, and $D = 32 - 36 = -4 < 0$, so that line misses. $k = -8$ is the height of the lowest point itself, where the line touches at $(1, -8)$, so -8 is correctly excluded by the strict inequality. $k = 0$ gives $x^2 - 2x - 7 = 0$ with $D = 32 > 0$, two crossings, as expected for a height above the minimum.

Matches Option **A**.

Why the other options are wrong

- **B** ($k < -6$): A sign slip on the correction term: adding instead of subtracting gives $y = (x-1)^2 + 1 - 7 = (x-1)^2 - 6$, a least value of -6 and the range $k < -6$. Substituting $x = 1$ into the original expression settles it, since $1 - 2 - 7 = -8$ and not -6. *(Sign error completing the square)*
- **C** ($k > -8$): The minimum -8 is found correctly and then the inequality is pointed the wrong way, treating heights above the lowest point as the ones the curve avoids. In fact every $k > -8$ gives $D = 32 + 4k > 0$ and two crossings, so this range describes the lines that cut the curve, not the lines that miss it. *(Inequality reversed)*
- **D** ($k > 8$): Moving k to the wrong side before forming the discriminant: from $x^2 - 2x - 7 + k = 0$ the discriminant becomes $D = 4 - 4(k-7) = 32 - 4k$, and $32 - 4k < 0$ delivers $k > 8$. Testing $k = 9$ against the curve exposes it, since a line nine units above the axis clearly cuts an upward parabola whose lowest point is at -8. *(Sign error rearranging before the discriminant)*
- **E** ($k < -7$): Completing the square but never subtracting the square of the halved coefficient: writing $y = (x-1)^2 - 7$ leaves the constant untouched at -7, so the least value looks like -7 and the range comes out as $k < -7$. The check kills it: at $k = -7.5$ the discriminant is $32 + 4(-7.5) = 2 > 0$, so that line does cross the curve twice. *(Incomplete completing the square)*
- **F** ($k > -6$): Quotes the discriminant as $b^2 + 4ac$ instead of $b^2 - 4ac$. From $x^2 - 2x - (7+k) = 0$, with $a = 1$, $b = -2$ and $c = -7 - k$, that gives $4 + 4(-7-k) = -24 - 4k$, and $-24 - 4k < 0$ rearranges to $k > -6$. The correct form gives $4 + 28 + 4k = 32 + 4k$, and $k = 0$ exposes the slip, since $x^2 - 2x - 7 = 0$ plainly has two roots and so needs a positive discriminant. *(Sign error in the discriminant formula)*
- **G** ($k < -11$): Completes the square with the whole coefficient instead of half of it, writing $y = (x-2)^2 - 2^2 - 7 = (x-2)^2 - 11$, so the least value looks like -11 and the range prints as $k < -11$. Halving is what makes the bracket reproduce the $-2x$ term, since $(x-2)^2$ expands to $x^2 - 4x + 4$ and not $x^2 - 2x + 4$. The discriminant kills it: at $k = -10$, $D = 32 + 4(-10) = -8 < 0$, so that line does miss the curve yet this range excludes it. *(Coefficient of x not halved)*
- **H** ($k \le -8$): Writes the no-real-roots condition as $D \le 0$ rather than $D < 0$, so $32 + 4k \le 0$ gives $k \le -8$. The equality case is exactly the one that has to go: $k = -8$ makes $D = 0$, a repeated root at $x = 1$, and the line $y = -8$ touches the curve at its vertex $(1, -8)$. Touching is meeting, so -8 itself is not a value the line may take. *(Tangency wrongly included)*

Common mistake. Locating the lowest point correctly at height -8 and then choosing the wrong side of it. Values of k above -8 are heights the curve does reach, and each such line cuts the curve twice; it is the heights below the minimum that are never attained. Reading the answer straight off the vertex without asking which side of it the line has to sit is what turns a correct -8 into a wrong inequality.

Takeaway. The Cheat Code: Once a curve is written as $y = (x-p)^2 + q$, its y-values are exactly $y \ge q$. A horizontal line $y = k$ then cuts it twice when $k > q$, touches it once when $k = q$, and misses it when $k < q$. Sorting the three cases by comparing k with q is faster and safer than forming the discriminant from scratch, and it tells you at once whether the inequality should be strict.

Time-saving tip. Speed Heuristic: You never need the roots, and you never need the full discriminant here. Halve the coefficient of x to get the bracket $(x-1)^2$, then subtract the square of that half from the constant: $-7 - 1 = -8$ is the vertex height. The answer is everything strictly below it, so scan the options for the one reading $k < -8$.

Question 8

MM1.7 · Algebra and functions · ★★☆☆☆ · 85 s

Each of the five functions below is defined for all real x. Which is one-to-one?

A. $f(x) = \sin x$
B. $f(x) = x^4 + x$
C. $f(x) = |x| - 4$
D. $f(x) = x^3 - 3x$
E. $f(x) = x^3 + 2x$
F. $f(x) = 2^x - x$
G. $f(x) = 2^{x^2}$
H. $f(x) = x^2 - 6x$

Answer E. Key idea

A mapping is one-to-one only when no two different inputs share an output, so a single pair of distinct inputs giving equal outputs settles the matter against a function immediately. Proving the positive case takes more care: a function that increases strictly across its whole domain can never repeat an output, because a larger input always produces a larger output.

Worked solution

1. Hunt for a repeated output

A single pair of distinct inputs with the same output is enough to rule a function out, so test the candidates rather than trying to prove anything first.

Option A: $f(3) = |3| - 4 = -1$ and $f(-3) = |-3| - 4 = -1$.

Option C: $f(0) = 0 - 0 = 0$ and $f(6) = 36 - 36 = 0$.

Option D: $\sin 0^\circ = 0$ and $\sin 180^\circ = 0$.

Option E: $f(1) = 1 - 3 = -2$ and $f(-2) = -8 + 6 = -2$.

Each of those four sends two different inputs to one output, so each is many-to-one.

2. Confirm the survivor is one-to-one

$f(x) = x^3 + 2x$ is the sum of x^3 and $2x$, and each of those grows strictly as x grows. A sum of strictly increasing functions is strictly increasing, so $x_1 < x_2$ forces $f(x_1) < f(x_2)$ and two inputs can never share an output.

Sanity check. Suppose $x_1^3 + 2x_1 = x_2^3 + 2x_2$. Subtracting and factorising gives $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 2) = 0$. The second bracket equals $\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{3}{4}x_2^2 + 2$, which is never smaller than 2, so the first bracket must vanish and $x_1 = x_2$. The mapping is one-to-one.

Matches Option **E**.

Why the other options are wrong

- **A** ($f(x) = \sin x$): The sine function repeats every 360°, and even inside one period $\sin 0^\circ = 0$ and $\sin 180^\circ = 0$. Infinitely many inputs therefore share each output value. *(Overlooked periodicity)*
- **B** ($f(x) = x^4 + x$): Reasons that x^4 on its own repeats outputs but the added x destroys the symmetry, so the sum must be one-to-one. The fold survives, merely shifted off the axis: $f(0) = 0 + 0 = 0$ and $f(-1) = (-1)^4 + (-1) = 1 - 1 = 0$, two different inputs with the same output. An even leading power forces a minimum, and any graph with a minimum turns back on itself. *(Assumed a linear term removes the fold)*

- **C** ($f(x) = |x| - 4$): The modulus folds the negative half of the number line onto the positive half, so inputs of equal size give identical outputs: $f(3) = 3 - 4 = -1$ and $f(-3) = 3 - 4 = -1$. Two inputs, one output. *(Ignored the reflection built into the modulus)*
- **D** ($f(x) = x^3 - 3x$): Treating every cubic as one-to-one because x^3 is. The $-3x$ term creates two turning points: $f(1) = 1 - 3 = -2$ and $f(-2) = -8 + 6 = -2$, so this graph repeats outputs. *(Over-generalised from $y = x^3$)*
- **F** ($f(x) = 2^x - x$): Takes it that anything built from 2^x inherits its strict increase and therefore cannot repeat an output. Close to the origin the $-x$ term wins: $f(0) = 2^0 - 0 = 1$ and $f(1) = 2^1 - 1 = 1$. One output, two inputs, so the mapping is many-to-one; the graph dips between $x = 0$ and $x = 1$ before the exponential takes over. *(Assumed the exponential forces strict increase)*
- **G** ($f(x) = 2^{x^2}$): Applies the rule that $y = a^x$ is one-to-one without checking what sits in the index. The index here is x^2, which already sends inputs of equal size to the same number: $f(2) = 2^4 = 16$ and $f(-2) = 2^4 = 16$. The exponential step is one-to-one, but the squaring that feeds it is not, and a many-to-one first stage is enough to make the whole mapping many-to-one. *(Even index inside the exponent overlooked)*
- **H** ($f(x) = x^2 - 6x$): Every quadratic is symmetric about its vertex. Factorising, $x^2 - 6x = x(x - 6)$, so $f(0) = 0$ and $f(6) = 0$: the outputs pair up about $x = 3$. *(Overlooked quadratic symmetry)*

Common mistake. Assuming that every cubic is one-to-one because $y = x^3$ is. A cubic carrying a negative multiple of x doubles back on itself: for $x^3 - 3x$, $f(1) = -2$ and $f(-2) = -2$, so it repeats outputs exactly as the quadratic and the modulus do.

Takeaway. The Cheat Code: One counterexample kills injectivity, so go looking for a repeated output before attempting any proof. Symmetry (a modulus, an even power) and repetition (the trigonometric functions) hand you the pair on sight, and for $x^3 - kx$ with $k > 0$ the pair $f(0) = f(\sqrt{k}) = 0$ ends it in one line.

Time-saving tip. Speed Heuristic: Sort the options by shape before substituting anything. The modulus, the quadratic and the sine all repeat outputs by inspection, which leaves only the two cubics, and the one with the negative x term is the one that turns back. Total work: a single pair of substitutions.

Question 9

MM5.2 · Exponentials and logarithms · ★★★☆☆ · 90 s

Let $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 1$ is a fixed base. Express $\log_a\left(\dfrac{9\sqrt{a}}{8}\right)$ in terms of p and q.

A. $3q - 2p + \frac{1}{2}$
B. $2q - 3p + 2$
C. $q^2 - p^3 + \frac{1}{2}$
D. $\frac{2q}{3p} + \frac{1}{2}$
E. $2q + 3p + \frac{1}{2}$
F. $2q - 3p + \frac{1}{2}$
G. $2q - 3p + 1$
H. $3p - 2q - \frac{1}{2}$

Answer F. Key idea

The laws of logarithms convert the arithmetic inside a logarithm into simpler arithmetic outside it: a product becomes a sum, a quotient becomes a difference, and an index becomes a multiplying factor. So

> an argument that is built from powers of a few known numbers can always be dismantled into multiples of the logarithms of those numbers. A surd is only a fractional index, and a logarithm of the base itself is 1, which together fix the contribution of any root of the base without needing its value.

Worked solution

1. Split the quotient

The subtraction law turns the division inside the logarithm into a subtraction outside it:

$$\log_a\left(\frac{9\sqrt{a}}{8}\right) = \log_a(9\sqrt{a}) - \log_a 8$$

2. Split the remaining product

$$\log_a(9\sqrt{a}) = \log_a 9 + \log_a \sqrt{a}$$

3. Bring every index down as a factor

$$\log_a 9 = \log_a 3^2 = 2\log_a 3 = 2q$$

$$\log_a 8 = \log_a 2^3 = 3\log_a 2 = 3p$$

$$\log_a \sqrt{a} = \log_a a^{1/2} = \frac{1}{2}\log_a a = \frac{1}{2}$$

4. Reassemble

$$\log_a\left(\frac{9\sqrt{a}}{8}\right) = 2q + \frac{1}{2} - 3p = 2q - 3p + \frac{1}{2}$$

Sanity check with a concrete base. Take $a = 2$, so $p = \log_2 2 = 1$ and the expression predicts $2q - 3 + \frac{1}{2} = 2q - \frac{5}{2}$. Working directly, $\log_2\left(\frac{9\sqrt{2}}{8}\right) = \log_2 9 + \log_2\sqrt{2} - \log_2 8 = 2\log_2 3 + \frac{1}{2} - 3 = 2q - \frac{5}{2}$, the same expression, so the dismantling is consistent.

Matches Option **F**.

Why the other options are wrong

- **A** ($3q - 2p + \frac{1}{2}$): Recognises the indices 2 and 3 but pairs them with the wrong logarithms, writing $\log_a 9 = 3q$ and $\log_a 8 = 2p$ to give $3q - 2p + \frac{1}{2}$. The multiplier comes from the index of the number itself: $9 = 3^2$ contributes $2q$ and $8 = 2^3$ contributes $3p$. *(Coefficients attached to the wrong letters)*
- **B** ($2q - 3p + 2$): Converts the surd with the index the wrong way up, taking $\log_a\sqrt{a} = 2\log_a a = 2$ instead of $\frac{1}{2}\log_a a = \frac{1}{2}$, so the total becomes $2q - 3p + 2$. A square root is the index $\frac{1}{2}$: $\sqrt{a}\times\sqrt{a} = a$ forces the two indices to add to 1, so each is one half, and the index 2 belongs to a^2. *(Root index inverted)*
- **C** ($q^2 - p^3 + \frac{1}{2}$): Raises the logarithm to the index instead of multiplying by it, taking $\log_a 3^2 = (\log_a 3)^2 = q^2$ and $\log_a 2^3 = (\log_a 2)^3 = p^3$, for $q^2 - p^3 + \frac{1}{2}$. The law reads $\log_a x^k = k\log_a x$, so the index becomes a factor in front, not an index on the answer. *(Power law applied to the logarithm)*
- **D** ($\frac{2q}{3p} + \frac{1}{2}$): Handles $\sqrt{a}$ correctly but writes $\log_a\frac{9}{8} = \frac{\log_a 9}{\log_a 8} = \frac{2q}{3p}$. Division inside a logarithm becomes subtraction outside it, so that part is $2q - 3p$; a ratio of two logarithms is a change of base, which is a different statement entirely. *(Quotient inside turned into a quotient outside)*

- **E** ($2q + 3p + \frac{1}{2}$): Applies the product law to a quotient, writing $\log_a\left(\frac{9\sqrt{a}}{8}\right) = \log_a 9 + \log_a \sqrt{a} + \log_a 8 = 2q + \frac{1}{2} + 3p$. Dividing inside a logarithm subtracts outside it, so the 8 contributes $-3p$. Testing $a = 2$, where $p = 1$, this reads $2q + \frac{7}{2}$ against the true $2q - \frac{5}{2}$, an error of 6, which is exactly $2\log_a 8$. *(Division inside turned into addition)*
- **G** ($2q - 3p + 1$): Treats $\sqrt{a}$ as though it were a itself, so the term contributes $\log_a a = 1$ and the total becomes $2q - 3p + 1$. Since $\sqrt{a} = a^{1/2}$, the index $\frac{1}{2}$ comes down as a factor and the term is worth $\frac{1}{2}$, so this exceeds the correct value by exactly $\frac{1}{2}$. *(Root of the base mishandled)*
- **H** ($3p - 2q - \frac{1}{2}$): Subtracts the numerator's logarithm from the denominator's, writing $\log_a 8 - \log_a(9\sqrt{a}) = 3p - \left(2q + \frac{1}{2}\right) = 3p - 2q - \frac{1}{2}$. That is the logarithm of $\frac{8}{9\sqrt{a}}$, the reciprocal of the argument asked for, so every term has the wrong sign. The quotient law reads top minus bottom. *(Subtraction taken in the wrong order)*

Common mistake. Turning the division inside the logarithm into a division of logarithms, writing $\log_a \frac{9}{8}$ as $\frac{\log_a 9}{\log_a 8}$. The law says the logarithm of a quotient is the difference of the logarithms, so the two terms are subtracted, never divided.

Takeaway. **The Cheat Code:** Rewrite every number inside the logarithm as a power of something you already have a letter for, then let each index step outside as a multiplier. Roots are just indices too, and any root of the base itself contributes only its fractional index.

Time-saving tip. Speed Heuristic: Deal with $\sqrt{a}$ first, because a root of the base always contributes its index alone, here $\frac{1}{2}$, whatever the base is. That fixes the constant immediately, and the only remaining question is which of 2 and 3 multiplies which letter: $9 = 3^2$ gives $2q$, and $8 = 2^3$ gives $3p$.

Question 10

MM4.5 · Trigonometry · ★★★★☆ · 90 s

For an acute angle θ, $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{25}{12}$. Find the exact value of $\tan\theta$.

A. $\frac{3}{4}$
B. $\frac{24}{25}$
C. $\frac{4}{3}$
D. $\frac{24}{7}$
E. $\frac{7}{24}$

Answer D. Key idea

The Pythagorean identity is what converts information about one trigonometric ratio into information about the others, since $\sin^2\theta + \cos^2\theta = 1$ ties them together for every angle. That means an expression built from sine and cosine can often be collapsed to a single ratio before any value is extracted, which is

> much cheaper than solving for both at once. Once one ratio is pinned down, the quadrant fixes the sign of the other, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ converts the pair into the tangent that was asked for.

Worked solution

1. Put the two fractions over a common denominator

For an acute angle neither $\sin\theta$ nor $1 + \cos\theta$ is zero, so the sum may be combined:

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

2. Collapse the numerator with the Pythagorean identity

Expanding the square and using $\sin^2\theta + \cos^2\theta = 1$:

$$\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta = 1 + 1 + 2\cos\theta = 2(1+\cos\theta)$$

3. Cancel and read off the sine

The factor $1 + \cos\theta$ appears above and below, so the whole left side is

$$\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta}$$

Hence $\dfrac{2}{\sin\theta} = \dfrac{25}{12}$, giving $\sin\theta = \dfrac{24}{25}$.

4. Recover the cosine, then the tangent

$$\cos^2\theta = 1 - \left(\frac{24}{25}\right)^2 = \frac{625-576}{625} = \frac{49}{625}$$

The angle is acute, so $\cos\theta = +\dfrac{7}{25}$ and

$$\tan\theta = \frac{24/25}{7/25} = \frac{24}{7}$$

Sanity check: with these values the first fraction is $\dfrac{24/25}{32/25} = \dfrac{3}{4}$ and the second is its reciprocal $\dfrac{4}{3}$, and $\dfrac{3}{4} + \dfrac{4}{3} = \dfrac{9+16}{12} = \dfrac{25}{12}$, the value given. The triple 7, 24, 25 also satisfies $49 + 576 = 625$, so the identity holds exactly.

Matches Option **D**.

Why the other options are wrong

- **A** ($\frac{3}{4}$): Evaluates the first fraction once $\sin\theta = \dfrac{24}{25}$ and $\cos\theta = \dfrac{7}{25}$ are known: $\dfrac{24/25}{1+7/25} = \dfrac{24/25}{32/25} = \dfrac{3}{4}$, and reports that. It is a genuine value in the problem, but it is $\dfrac{\sin\theta}{1+\cos\theta}$, not $\dfrac{\sin\theta}{\cos\theta}$. *(Intermediate fraction quoted as the answer)*
- **B** ($\frac{24}{25}$): Solves $\dfrac{2}{\sin\theta} = \dfrac{25}{12}$ to get $\sin\theta = \dfrac{24}{25}$ and stops there. The sine is only the halfway point: the cosine still has to come from $\cos^2\theta = 1 - \dfrac{576}{625} = \dfrac{49}{625}$ before the ratio $\dfrac{24}{7}$ can be formed. *(Answers the wrong quantity)*

- **C** ($\frac{4}{3}$): Uses $1 - \cos\theta$ in place of $\cos\theta$: $\frac{24/25}{1 - 7/25} = \frac{24/25}{18/25} = \frac{4}{3}$. The tangent divides by the cosine itself, $\frac{7}{25}$, which gives $\frac{24}{7}$. *(Wrong denominator used for the tangent)*
- **E** ($\frac{7}{24}$): Reads the collapsed left side as $\frac{2}{\cos\theta}$, so $\cos\theta = \frac{24}{25}$ and $\sin\theta = \frac{7}{25}$, giving $\tan\theta = \frac{7}{24}$. The factor that cancels is $1 + \cos\theta$, leaving $\sin\theta$ alone in the denominator, so it is the sine that equals $\frac{24}{25}$. *(Sine and cosine interchanged)*

Common mistake. Expanding $(1 + \cos\theta)^2$ as $1 + \cos^2\theta$ and losing the cross term $2\cos\theta$. Without it the numerator becomes the constant 2, nothing cancels, and the equation turns into a messy one in two unknowns instead of collapsing to $\frac{2}{\sin\theta}$.

Takeaway. The Cheat Code: When a sine and cosine expression is a fraction plus its own reciprocal, combine before you substitute anything. The cross term from the square is exactly what $\sin^2\theta + \cos^2\theta = 1$ needs to produce a repeated factor, and that factor cancels the whole problem down to one ratio.

Time-saving tip. Speed Heuristic: The second fraction is the reciprocal of the first, and combining always leaves $\frac{2}{\sin\theta}$, so $\sin\theta = \frac{24}{25}$ can be written down in one line. Then recognise 7, 24, 25 as a Pythagorean triple rather than taking a square root, and the tangent is immediate.

END OF SOLUTIONS

Questions 11 to 27 are in the full paper

This is an extract. The 10 questions before this page are real questions from a commissioned worked solutions, shown in full and with nothing withheld, so that the standard can be judged rather than described.

The rest of the paper is not hidden behind this page. It was never in this file. Each commissioned paper is written for one school, recorded against it, and issued to no other, so the questions in it are not published anywhere, including here.

A commissioned paper is NOT this paper with the missing questions restored. Every question in it is different from every question here, the 10 shown above included. Your school's paper is drawn fresh, and no question that has ever been published, on this page or anywhere else, can appear in it.

A full paper runs to 27 questions in this module alone, with a question paper, an answer sheet, a mark scheme and a worked solution for every question, across all five ESAT modules.

Commission a paper for your school

lucasrey.com/teaching/for-schools/

Written by researchers in engineering, physics and mathematics at Oxford and Cambridge, and checked question by question before it is issued.

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/

This page is in the full paper.

lucasrey.com/teaching/for-schools/