EXAMPLE GRAMMAR SCHOOL

ESAT Mock: Worked Solutions

Mathematics 1

Paper SMP-2026-09 · Version 1.5

Staff copy

This document contains the answers. Keep it away from candidates until the paper has been sat. Marking guidance is in the separate mark scheme.

Question 1

M1.1 · Units and compound measures · ★★☆☆☆ · 85 s

A winch unwinds 0.75 km of uniform steel cable of linear density 12 g cm^{-1}. Find its mass.

- **A.** 900 kg
- **B.** 90 kg
- **C.** 625 kg
- **D.** 9 kg
- **E.** 90000 kg
- **F.** 9000 kg
- **G.** 0.9 kg
- **H.** 0.09 kg

Answer A. Key idea

A compound unit is a recipe for a division, and every unit inside it has to be rescaled separately before the number in front of it means anything. A linear density written in grams per centimetre carries a mass unit on the top and a length unit on the bottom, so moving it to kilograms per metre multiplies by one factor and divides by another. Once the rate is expressed in units that match the length being measured, the mass is a single multiplication, because mass equals rate multiplied by length.

Worked solution

1. Rewrite the linear density in kg m^{-1}

A linear density of 12 g cm^{-1} says that every centimetre of cable weighs 12 g. A metre contains 100 of those centimetres, so one metre of cable weighs

$12 \times 100 = 1200$ g

and 1200 g = 1.2 kg, so the cable is 1.2 kg m^{-1}.

Both unit changes are doing work here and they pull in opposite directions: the length unit contributes a factor of 100 on the bottom, the mass unit a factor of $\frac{1}{1000}$ on the top, and the survivor is a net division by 10. That is why 12 becomes 1.2 and not 1200 or 0.012.

2. Convert the length and multiply

0.75 km = $0.75 \times 1000 = 750$ m, and the rate is now per metre, so the two agree:

$m = 1.2 \times 750 = 750 + 150 = 900$ kg

Check: run it backwards. 900 kg spread over 750 m is 1.2 kg per metre, which is 1200 g per metre, which is 12 g per centimetre, exactly the figure given. A three quarter kilometre of heavy steel cable weighing close to a tonne is also the right order of magnitude for a winch drum.

Matches Option **A**.

Why the other options are wrong

- **B** (90 kg): Took 0.75 km as 75 m, using a factor of 100 for kilometres to metres instead of 1000: $1.2 \times 75 = 90$. A kilometre is a thousand metres, and the factor of 100 belongs to centimetres. *(Wrong power of ten for km)*
- **C** (625 kg): Reached 1.2 kg m^{-1} and converted the length correctly, then divided by the rate instead of multiplying by it: $750 \div 1.2 = 7500 \div 12 = 625$. A rate in kilograms per metre multiplies a number of metres; dividing a number of metres by it leaves square metres per kilogram, which is not a mass at all. *(Rate inverted, divided instead of multiplied)*
- **D** (9 kg): Converted the mass but not the length: 12 g = 0.012 kg, then $0.012 \times 750 = 9$. That multiplies a per centimetre rate by a number of metres, so the answer is short by the factor of 100 that the length unit carries. *(Half a compound unit converted)*
- **E** (90000 kg): Knew that a single factor of ten survives between g cm^{-1} and kg m^{-1} but multiplied by it instead of dividing, reading 12 g cm^{-1} as 120 kg m^{-1}, then $120 \times 750 = 90000$. Dividing by 1000 for the mass outweighs multiplying by 100 for the length, so the net move is a division and the rate is 1.2, not 120. *(Net factor applied the wrong way)*
- **F** (9000 kg): Read the numeral 12 as though it were already 12 kg m^{-1} and multiplied: $12 \times 750 = 9000$. Neither the gram nor the centimetre was rescaled, so the figure is ten times the true rate. *(No conversion at all)*
- **G** (0.9 kg): Reached 1.2 kg m^{-1} correctly but multiplied by 0.75 instead of 750, leaving the length in kilometres: $1.2 \times 0.75 = 0.9$. The rate is per metre, so the length must be in metres. *(Length unit not converted)*
- **H** (0.09 kg): Divided for both unit changes: $12 \div 1000 = 0.012$ kg cm^{-1}, then divided again by 100 for centimetres to metres to give 0.00012 kg m^{-1}, and $0.00012 \times 750 = 0.09$. The length unit sits on the bottom of the rate, so going from a per centimetre rate to a per metre rate multiplies by 100. *(Both conversions taken as divisions)*

Common mistake. Rescaling the mass unit and leaving the length unit alone, or the other way round. Turning 12 g cm^{-1} into 0.012 kg cm^{-1} and then multiplying straight by 750 treats a per centimetre rate as a per metre rate, and the answer comes out a hundred times too small. Both halves of a compound unit have to be converted before the number is used.

Takeaway. The Cheat Code: 1 g cm^{-1} = 0.1 kg m^{-1}, because dividing by 1000 for the mass and multiplying by 100 for the length leaves a single division by 10. Fix the rate first, fix the length second, then multiply once.

Time-saving tip. Speed Heuristic: do not chase the powers of ten one at a time. Grams to kilograms is $\div 1000$ and per centimetre to per metre is $\times 100$, a net $\div 10$, so 12 g cm^{-1} is 1.2 kg m^{-1} on sight. Then 1.2×750 is 750 plus a fifth of 750, that is $750 + 150 = 900$.

Question 2

M5.18 · Geometry · ★★★☆☆ · 90 s

A ski tow climbs a uniform slope in a straight line. The cable rises 1.5 m for every 2 m of horizontal distance. Find the height gained along 20 m of cable.

- **A.** 15 m
- **B.** 12 m
- **C.** 6 m
- **D.** 16 m
- **E.** 8 m

Answer B. Key idea

For a fixed angle the three ratios sin, cos and tan are fixed numbers, so a slope quoted as a vertical rise per horizontal run hands you tan directly. A distance measured along the slope itself is the hypotenuse, not the adjacent side, so the ratio that connects it to the height is sin. Once the two given lengths are reduced to the 3, 4, 5 side lengths, both ratios are available and the whole conversion is exact arithmetic.

Worked solution

Step 1: Build the right-angled triangle

The cable, the horizontal ground and the vertical rise form a right-angled triangle. Taking θ as the angle of climb at the bottom of the slope, the rise is the side opposite θ and the horizontal distance is the side adjacent to θ, so opposite = 1.5 m and adjacent = 2 m.

Step 2: Read off the ratio the slope statement gives

TOA: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{1.5}{2} = \dfrac{3}{4}$.

Step 3: Switch to the ratio that matches the given length

The 20 m is measured along the cable, which is the hypotenuse, so the height needs sin, not tan. From opposite : adjacent = 3 : 4 the hypotenuse is $\sqrt{3^2+4^2} = 5$, so the sides are in the ratio 3 : 4 : 5 and $\sin\theta = \dfrac{3}{5}$.

Step 4: Apply sin to the cable length

SOH: height $= 20 \times \dfrac{3}{5} = 12$ m.

Sanity check: the same 20 m of cable covers $20 \times \dfrac{4}{5} = 16$ m horizontally, and $\dfrac{12}{16} = \dfrac{3}{4}$, which is the $\tan\theta$ we started from. The height is also comfortably less than the cable length, as it must be.

Matches Option **B**.

Why the other options are wrong

- **A** (15 m): Uses tan on the cable length: $20 \times \dfrac{3}{4} = 15$ m. That is the height only if the 20 m were the horizontal distance, but it is measured along the slope, so it is the hypotenuse. *(Conceptual Error)*
- **C** (6 m): Mixes the ratio numbers with the measured lengths: takes the hypotenuse as 5 from the 3 : 4 : 5 ratio while keeping the rise as the raw 1.5 m, giving $\sin\theta = \dfrac{1.5}{5} = 0.3$ and $20 \times 0.3 = 6$ m. The ratio must be built from two lengths of the same triangle. *(Scaling Error)*
- **D** (16 m): Picks CAH instead of SOH: $20 \times \dfrac{4}{5} = 16$ m. That is the horizontal distance the tow covers, not the height it gains. *(Conceptual Error)*

- **E** (8 m): Stops at the scale factor. The small triangle has hypotenuse 2.5 m, so $20 \div 2.5 = 8$, and the 8 is reported as the answer instead of being used to scale the rise: $8 \times 1.5 = 12$ m. *(Incomplete Solution)*

Common mistake. Treating the 20 m of cable as a horizontal distance and multiplying by $\tan\theta$, giving $20 \times \frac{3}{4} = 15$ m. The 20 m runs along the slope, so it is the hypotenuse and sin is the ratio that applies.

Takeaway. The Cheat Code: Before choosing a ratio, decide which side of the triangle the given number actually is. A rise per horizontal run fixes tan, but any distance measured along the slope is a hypotenuse, so the ratio that turns it into a height is sin.

Time-saving tip. Speed Heuristic: 1.5 : 2 scales to 3 : 4, so this is a 3, 4, 5 triangle. For any length measured along the slope, divide by 5 and multiply by 3: $20 \div 5 = 4$, then $4 \times 3 = 12$, with no fractions written down.

Question 3

M2.12 · Number · ★★★☆☆ · 90 s

240 g of alloy occupies 30 cm^3, each to the nearest ten. Find the greatest density.

- **A.** 9.4 g cm^{-3}
- **B.** 9.8 g cm^{-3}
- **C.** 8 g cm^{-3}
- **D.** 10 g cm^{-3}
- **E.** 7 g cm^{-3}
- **F.** 9.6 g cm^{-3}
- **G.** 12 g cm^{-3}
- **H.** 12.5 g cm^{-3}

Answer B. Key idea

A figure rounded to the nearest ten is only guaranteed to lie within half of that ten, so each measurement becomes an interval five units wide on either side of the stated value. Density is a quotient, and a quotient is at its largest when the numerator is at its largest and the denominator at its smallest, so the two measurements must be pushed to opposite ends of their intervals rather than to the same end.

Worked solution

1. Turn each rounding into an interval

A value given to the nearest ten carries half of ten, that is 5, of slack on either side. Writing m for the mass and V for the volume,

$$235 \le m < 245 \text{ and } 25 \le V < 35.$$

The two intervals are the same width in absolute terms, but the volume interval is far wider as a proportion of its measurement, which is why the volume controls the answer.

2. Choose the ends that maximise a quotient

Density is $\rho = \frac{m}{V}$. A quotient grows when the numerator grows and when the denominator shrinks, so the greatest possible density pairs the heaviest allowed mass with the smallest allowed volume:

$$\rho_{\max} = \frac{245}{25}.$$

3. Evaluate and check

$$\frac{245}{25} = \frac{490}{50} = \frac{49}{5} = 9.8 \text{ g cm}^{-3}.$$

Sanity check: the stated figures give $240 \div 30 = 8\ \mathrm{g\,cm^{-3}}$, and the smallest allowed density is $235 \div 35 = 6.71\ldots\ \mathrm{g\,cm^{-3}}$. An upper bound has to sit above the nominal value and it does, since $9.8 > 8$.

Matches Option **B**.

Why the other options are wrong

- **A** ($9.4\ \mathrm{g\,cm^{-3}}$): Paired the two lower bounds: $235 \div 25 = 9.4$. The numerator has to be at its upper bound for the quotient to be greatest. *(Both lower bounds used)*
- **C** ($8\ \mathrm{g\,cm^{-3}}$): Divided the stated figures and ignored the rounding altogether: $240 \div 30 = 8$. That is the nominal density, which sits inside the interval rather than at its top. *(Bounds ignored)*
- **D** ($10\ \mathrm{g\,cm^{-3}}$): Gave the mass the full ten grams of slack instead of half of it: $250 \div 25 = 10$. The nearest ten guarantees only 5 either way, so 250 is outside the possible range. *(Whole quantum instead of half)*
- **E** ($7\ \mathrm{g\,cm^{-3}}$): Pushed both measurements to the top of their intervals, as though the greatest density meant the largest number everywhere: $245 \div 35 = 7$. The denominator has to be at its smallest for a quotient to be greatest, so the volume belongs at 25, not 35. *(Both upper bounds used)*
- **F** ($9.6\ \mathrm{g\,cm^{-3}}$): Took the volume down to its lower bound but left the mass at the figure printed in the stem: $240 \div 25 = 9.6$. The mass is rounded to the nearest ten as well, so it carries the same five grams of slack and must go up to 245. *(Only the denominator moved)*
- **G** ($12\ \mathrm{g\,cm^{-3}}$): Gave the volume the full ten of slack while leaving the mass alone: $240 \div 20 = 12$. Rounding to the nearest ten guarantees only five either way, so the volume cannot fall below 25, and the mass is free to rise to 245. *(Whole quantum on the volume, mass unmoved)*
- **H** ($12.5\ \mathrm{g\,cm^{-3}}$): Used the full ten of slack on both measurements: $250 \div 20 = 12.5$. Both intervals are twice as wide as the rounding allows. *(Whole quantum instead of half)*

Common mistake. Pushing both measurements to the same end of their intervals, as though "largest" meant taking the largest number everywhere. For a quotient the two ends must be opposite: 245 over 25, not 245 over 35 and not 235 over 25.

Takeaway. **The Cheat Code:** For the maximum of a quotient take the top of the numerator's interval over the bottom of the denominator's, and get each interval from half of the stated rounding unit, so nearest ten means five either way.

Time-saving tip. Speed Heuristic: Only one pairing needs evaluating, so write $245 \div 25$ straight down and double both parts to $490 \div 50 = 9.8$. There is no need to work out the nominal or minimum density first.

Question 4

M4.11 · Algebra · ★★★☆☆ · 90 s

A start-up's cash balance C, in thousands of pounds, at the end of month x is modelled by $C = (x-1)(x-5)$, valid for months 0 to 6. Treating x as continuous, what is the lowest balance predicted?

A. $-\$3000$
B. $\$0$
C. $-\$9000$
D. $\$5000$
E. $-\$4000$
F. $\$1000$
G. $-\$14000$
H. $\$3000$

Answer E. Key idea

A quadratic with a positive squared term has a single minimum, and completing the square puts it in the form $(x - p)^2 + q$, where the bracket cannot be negative, so the least value is q and it occurs at $x = p$. The brackets of a factorised quadratic give the values where the curve is zero, not where it is least; the minimum sits exactly midway between them because the curve is symmetric about its turning point.

Worked solution

1. Put the model in standard form

$$C = (x - 1)(x - 5) = x^2 - 6x + 5$$

2. Complete the square

$$x^2 - 6x + 5 = (x - 3)^2 - 9 + 5 = (x - 3)^2 - 4$$

3. Read off the turning point

$(x - 3)^2$ is never negative and is zero only at $x = 3$, so the least value of C is -4, reached in month 3. The same x comes from $-\dfrac{b}{2a} = \dfrac{6}{2} = 3$, and substituting back gives $9 - 18 + 5 = -4$.

4. Check the range, then convert the units

Month 3 lies inside the valid span of months 0 to 6, so the model is being used where it applies. Either side, $C(2) = (1)(-3) = -3$ and $C(4) = (3)(-1) = -3$, both above -4, which confirms $x = 3$ as the trough. Since C is in thousands of pounds, the lowest balance is $-\$4000$.

Matches Option **E**.

Why the other options are wrong

- **A** ($-\$3000$): The midpoint of the roots was found as half their difference, $(5 - 1) \div 2 = 2$, rather than half their sum. Then $C(2) = (2 - 1)(2 - 5) = -3$, giving $-\$3000$, which is a real balance on the curve but not the least one. *(Midpoint from the difference)*
- **B** ($\$0$): Read the factorised form as handing the answer over: $C = 0$ at $x = 1$ and at $x = 5$, and the balance there was quoted as the lowest. Those are the two months the balance passes through zero, and the curve dips below zero between them, reaching $(3 - 1)(3 - 5) = -4$ thousand in month 3. *(Balance at a root taken as the minimum)*
- **C** ($-\$9000$): The square was completed to $(x - 3)^2 - 9 + 5$ and the working stopped at the -9, so the $+5$ from the original constant was never carried through. That reports $-\$9000$ instead of -4 thousand. *(Constant not carried through)*
- **D** ($\$5000$): The opening balance was taken as the lowest point: $C(0) = (0 - 1)(0 - 5) = 5$, that is $\$5000$. The curve falls after month 0, so its starting value is the highest point in the first half, not the lowest. *(Endpoint taken as minimum)*
- **F** ($\$1000$): The brackets were expanded as $x^2 - 4x + 5$ by subtracting the roots instead of adding them. That puts the turning point at $x = 2$ with value $4 - 8 + 5 = 1$, giving $\$1000$. Expanding correctly gives $-6x$, since $-1 - 5 = -6$. *(Expansion sign slip)*
- **G** ($-\$14000$): Expanded $(x - 1)(x - 5)$ as $x^2 - 6x - 5$, taking $(-1) \times (-5)$ as -5. Completing the square on that gives $(x - 3)^2 - 9 - 5 = (x - 3)^2 - 14$, so the minimum reads as -14 thousand. Two negatives multiply to $+5$, and $-9 + 5 = -4$. *(Sign slip on the product of the constants)*

- **H** ($3000): Located the turning point correctly, since $(x-3)^2-4$ is least at $x=3$, then reported the 3 instead of the value there. The 3 is the month the trough falls in; substituting it gives $C=(3-1)(3-5)=-4$, that is −$4000. *(Month of the minimum reported as the balance)*

Common mistake. Quoting a root, or the balance at a root, as the lowest point. The factors give $x=1$ and $x=5$, where the balance is zero, and zero is higher than the trough between them. Roots answer where the curve crosses, never how low it goes.

Takeaway. The Cheat Code: For a factorised quadratic the turning point sits midway between the roots, so average them and substitute once. Completing the square says the same thing: the bracket vanishes at the midpoint and the number left outside is the minimum.

Time-saving tip. Speed Heuristic: average the roots, $(1+5)\div 2=3$, then feed that straight back into the brackets already given: $(3-1)(3-5)=2\times(-2)=-4$. No expansion, no formula and no completing the square are needed.

Question 5

M3.8 · Ratio and proportion · ★★☆☆☆ · 45 s

Decrease 80 by 15%.

A. 92
B. 68
C. 65
D. 94.12
E. 12

Answer B. Key idea

Percentage increases and decreases are handled fastest with a single decimal multiplier. A 15% decrease leaves 85% of the original, so the multiplier is 0.85 and $80\times 0.85=68$.

Worked solution

1. Choose the multiplier

A decrease of 15% leaves 100% − 15% = 85% of the original amount, so the multiplier is 0.85.

2. Apply it in one step

$$80\times 0.85=68$$

Doing it in two steps (15% of $80=12$, then $80-12=68$) gives the same value, but the single multiplier is quicker and leaves nothing to forget.

3. Sense-check the direction

The question asks for a decrease, so the answer must be smaller than 80. That rules out 92 and 94.12 before any arithmetic is checked.

4. Why each of the other options fails

- **A** 92: applied the change the wrong way, 80×1.15, giving an increase.
- **C** 65: subtracted 15 as a raw number rather than 15% of 80, which is 12.
- **D** 94.12: worked backwards, $80\div 0.85$, as if 80 were the value after the decrease.
- **E** 12: found 15% of 80 but never took it away from the original.

Matches Option **B**.

Why the other options are wrong

- **A** (92): Applied the opposite change. *(Conceptual Error)*
- **C** (65): Added/subtracted the percentage as a raw number. *(Conceptual Error)*
- **D** (94.12): Calculated original value before change (reverse percentage). *(Conceptual Error)*
- **E** (12): Calculated the percentage, but didn't apply the change. *(Conceptual Error)*

Common mistake. Forgetting to add/subtract from the original.

Takeaway. The Cheat Code: Do not calculate the percentage and then add/subtract it in two separate steps. Use a single multiplier (e.g., Original × 0.85 = New for a 15% decrease). For reverse percentages, NEVER add the percentage back on to the new amount; instead, divide by the multiplier (New / 0.85 = Original).

Time-saving tip. Decide first whether the quantity scales up or down, then check your answer against that expectation before you look at the options.

Question 6

M5.16 · Geometry · ★★★☆☆ · 90 s

A circular ornamental garden has a diameter of 36 m, with a sundial at its centre. A wedge-shaped vegetable patch has its point at the sundial and reaches out to the garden's edge, and it covers 54π m^2 of the ground. Rabbit fencing is fitted along the curved edge of the patch and nowhere else. What length of the garden's boundary is left unfenced?

A. 33π m
B. 36π m
C. 60π m
D. 42π m
E. 69π m
F. 30π m
G. 15π m
H. 6π m

Answer F. Key idea

A sector is a fixed fraction of its circle, and the single fraction $\frac{\theta}{360}$ governs the sector's area, its arc and its angle alike. A patch that takes one sixth of the disc therefore has one sixth of the circumference as its curved edge, whether or not the angle is ever written down. When an area is given and a length is wanted, that fraction is the bridge: read it off by comparing the sector area with the area of the whole circle, then apply it to the circumference. The part of the boundary outside the sector takes the complementary share of the same circumference.

Worked solution

Step 1: Halve the diameter and size the whole garden

The garden is 36 m across, so the radius is $r = 18$ m and the whole circle covers

$\pi r^2 = \pi \times 18^2 = 324\pi$ m^2.

Step 2: Read off the share taken by the patch

The patch is a sector, so its area is $\frac{\theta}{360}$ of the disc:

$\frac{\theta}{360} \times 324\pi = 54\pi$, so $\frac{\theta}{360} = \frac{54}{324} = \frac{1}{6}$.

Step 3: Apply the same share to the boundary

The boundary of the garden has length $2\pi r = 2\pi \times 18 = 36\pi$ m, and the patch's curved edge is the same sixth of it:

$\frac{1}{6} \times 36\pi = 6\pi$ m carries fencing.

Step 4: Remove the fenced arc

$36\pi - 6\pi = 30\pi$ m of boundary is left unfenced.

Check: 30π is five sixths of 36π, which is the share of the boundary lying outside a patch that takes one sixth of the garden. Numerically the answer is about 94 m out of a full boundary of about 113 m, so the fenced piece is the small part, as a one sixth share must be.

Matches Option **F**.

Why the other options are wrong

- **A** (33π m): Slipped the arc formula's factor of 2π into the area, using $\frac{\theta}{360} \times 2\pi r^2 = 54\pi$. That gives $\frac{\theta}{360} = \frac{54}{648} = \frac{1}{12}$, an arc of $\frac{1}{12} \times 36\pi = 3\pi$, and $36\pi - 3\pi = 33\pi$ m. A sector's area is $\frac{\theta}{360}\pi r^2$. *(Area formula confusion)*
- **B** (36π m): Gave the whole boundary, $2\pi \times 18 = 36\pi$ m, without removing the 6π m that the rabbit fencing occupies. *(Subtraction omitted)*
- **C** (60π m): Halved the 36 m for the area, so $\pi \times 18^2 = 324\pi$ and the share is $\frac{54}{324} = \frac{1}{6}$, then reached for the printed 36 again in the length step: $2\pi \times 36 = 72\pi$. That makes the fenced arc $\frac{1}{6} \times 72\pi = 12\pi$ and leaves $72\pi - 12\pi = 60\pi$ m. The radius is 18 m in every step, not just the first. *(Diameter reused as the radius in the boundary step)*
- **D** (42π m): Both lengths were right, the boundary $2\pi \times 18 = 36\pi$ m and the fenced arc $\frac{1}{6} \times 36\pi = 6\pi$ m, but they were combined the wrong way: $36\pi + 6\pi = 42\pi$ m. The patch reaches the garden's edge, so its curved edge is part of the boundary and has to come off it, giving $36\pi - 6\pi = 30\pi$ m. *(Arc added instead of removed)*
- **E** (69π m): Took the 36 m as the radius. The circle then appears to be $\pi \times 36^2 = 1296\pi$ m^2, so the patch looks like $\frac{54}{1296} = \frac{1}{24}$ of it, and $\frac{1}{24}$ of the inflated boundary 72π is 3π, leaving $72\pi - 3\pi = 69\pi$ m. *(Diameter used as radius)*
- **G** (15π m): Found the share correctly, $\frac{54\pi}{324\pi} = \frac{1}{6}$, then measured the boundary with $\pi r = 18\pi$ m instead of $2\pi r = 36\pi$ m. The fenced arc then looks like $\frac{1}{6} \times 18\pi = 3\pi$, and $18\pi - 3\pi = 15\pi$ m. The circumference is πd, that is $2\pi r$, so a 36 m garden has a 36π m boundary. *(Circumference taken as πr)*
- **H** (6π m): This is the fenced arc itself, $\frac{1}{6} \times 36\pi = 6\pi$ m. The question asks for the length the fencing does not cover, which is the other five sixths of the boundary. *(Answered the complement)*

Common mistake. Using the 36 m as the radius because it is the length printed in the stem. It is the diameter, so the disc is 324π m^2 rather than 1296π m^2 and the boundary is 36π m rather than 72π m. The slip corrupts both the fraction and the circumference, and it lands on 69π m.

Takeaway. The Cheat Code: In a sector, the share of the area, the share of the arc and the share of the full turn are all the same fraction. Find that fraction once, apply it to whichever quantity the question wants, and if the question asks for the rest of the circle, use one minus the fraction instead of recomputing.

Time-saving tip. Speed Heuristic: cancel π on sight, so the areas give $\frac{54}{324} = \frac{1}{6}$ immediately. The unfenced boundary is then five sixths of 36π, that is 30π, in one multiplication, and the angle of $60°$ never has to be written down at all.

Question 7

M4.12 · Algebra · ★★★☆☆ · 90 s

A camera's light meter plots the correct exposure time t, in milliseconds, against the aperture area A, in mm^2. The plotted curve is a hyperbola in the first quadrant, falling steeply and approaching both axes without ever meeting them. One recorded setting has $A = 8$ and $t = 45$. What exposure time does the curve give when $A = 12$?

- **A.** 41 ms
- **B.** 22.5 ms
- **C.** 20 ms
- **D.** 101.25 ms
- **E.** 90 ms
- **F.** 3.75 ms
- **G.** 67.5 ms
- **H.** 30 ms

Answer H. Key idea

A hyperbola that lies in the first quadrant and has both axes as asymptotes is the graph of the reciprocal function, $t = \frac{k}{A}$. Its defining property is that the product of the two coordinates is the same at every point, so one recorded pair fixes k and every other point then follows by a single division. Because k is not zero, neither coordinate can ever reach zero, which is exactly why the curve closes on the axes without touching them.

Worked solution

1. Name the curve from its description

A hyperbola in the first quadrant with both axes as asymptotes is the reciprocal graph, so the meter is using

$$t = \frac{k}{A}, \qquad \text{that is} \qquad tA = k$$

2. Fix the constant from the recorded setting

$$k = 8 \times 45 = 360$$

3. Substitute the new area

$$t = \frac{360}{12} = 30\,\text{ms}$$

4. Check it against the shape of the curve

The area rose, and the curve falls, so the time must come out below 45 ms, and 30 does. The product test also passes: $12 \times 30 = 360$, the same as 8×45. Finally the value is positive and finite, as it must be for a curve that never reaches either axis.

Matches Option **H**.

Why the other options are wrong

- **A** (41 ms): The sum was held constant instead of the product: $8 + 45 = 53$, then $53 - 12 = 41$. A constant sum is a straight line of gradient -1, which would cut the A axis at 53 rather than run alongside it. *(Constant sum instead of product)*
- **B** (22.5 ms): The area rose by half, from 8 to 12, so the time was cut by half: $45 \times 0.5 = 22.5$. Under $tA = k$ the time is scaled by the ratio of the areas, $\frac{8}{12} = \frac{2}{3}$, which is a cut of a third and not a half. The product test also fails, since $12 \times 22.5 = 270$ rather than 360. *(Percentage change mirrored)*
- **C** (20 ms): An inverse square rule was used instead of the reciprocal: $45 \times \left(\frac{8}{12}\right)^2 = 45 \times \frac{4}{9} = 20$. That curve does approach both axes, but it is not a hyperbola and it fails the constant product test, since $12 \times 20 = 240$. *(Inverse square used)*
- **D** (101.25 ms): The area ratio was squared and applied the way the area grows: $45 \times \left(\frac{12}{8}\right)^2 = 45 \times 2.25 = 101.25$. That is the curve $t \propto A^2$, which climbs away from both axes rather than closing on them, and it fails the constant product test since $12 \times 101.25 = 1215$. *(Direct square proportion)*
- **E** (90 ms): The constant was found correctly, $k = 8 \times 45 = 360$, but then divided by the increase in area instead of the area itself: $\frac{360}{12-8} = 90$. The curve gives the time at $A = 12$, so the division is $\frac{360}{12} = 30$, and a wider aperture cannot need a longer exposure than the 45 ms already recorded. *(Divided by the change in area)*
- **F** (3.75 ms): The recorded time was divided by the new area, $45 \div 12 = 3.75$. The division must be applied to the constant 360, not to a time; dividing a time by an area does not even leave a time. *(Divided the wrong quantity)*
- **G** (67.5 ms): The time was scaled up with the area, $45 \times \frac{12}{8} = 67.5$, which is direct proportion. That is a straight line through the origin, and it contradicts a curve that falls as the area grows. *(Direct proportion assumed)*

Common mistake. Treating a falling curve as a falling straight line. A straight line through the recorded point would cross the A axis at some finite area, which would mean a real setting needing no exposure time at all. The stated asymptotes rule that out, and the stated hyperbola then forces the reciprocal model.

Takeaway. The Cheat Code: A hyperbola with asymptotes on both axes means the product of the coordinates is constant. Multiply the pair you are given, divide by the new value, and you are done in two operations.

Time-saving tip. Speed Heuristic: skip the constant. The area is multiplied by $\frac{12}{8} = \frac{3}{2}$, so on a reciprocal curve the time is multiplied by $\frac{2}{3}$, and two thirds of 45 is 30.

Question 8

M2.10 · Number · ★★☆☆☆ · 90 s

A battery is $\frac{3}{4}$ charged. Using 0.35 of its capacity leaves what percentage?

A. 40%
B. 110%
C. 48.75%
D. 60%
E. 74.65%
F. 71.5%
G. 65%
H. 26.25%

Answer A. Key idea

A fraction, a decimal and a percentage are three notations for the same kind of number, and they can only be combined once they are all written against the same reference quantity. Here both the charge held and the amount used are measured against the battery's full capacity, so converting each to a percentage of that capacity makes the calculation a single subtraction rather than a mixture of unlike quantities.

Worked solution

1. Write the starting charge as a percentage

A percentage is a number of parts per hundred, so multiply the fraction by 100:

$$\frac{3}{4} = \frac{3}{4} \times 100\% = 75\%.$$

The quarter is 25%, and three of them give 75%, so the battery starts at 75% of capacity.

2. Write the amount used as a percentage of the same thing

$$0.35 = \frac{35}{100} = 35\%.$$

This is stated as a share of the capacity, the same reference as the 75%, so the two figures can now be compared directly. Had it been a share of the charge present it would have had to be multiplied by the 75% instead.

3. Subtract and check

$75\% - 35\% = 40\%.$

Sanity check in decimals: $0.75 - 0.35 = 0.40$, which is 40% of capacity. The answer is below the starting 75% and above zero, as it must be, and the battery is left a little under half charged.

Matches Option **A**.

Why the other options are wrong

- **B** (110%): Added the amount used instead of subtracting it: $75\% + 35\% = 110\%$, a charge above full capacity, which is impossible. *(Wrong operation)*
- **C** (48.75%): Read the 0.35 as a share of the charge present rather than of the capacity, so 35% of 75% was removed: $75 \times 0.65 = 48.75$, giving 48.75%. *(Wrong reference quantity)*
- **D** (60%): Answered how much of the capacity is empty rather than how much charge remains: the battery starts $100\% - 75\% = 25\%$ empty and a further 35% is used, so $25 + 35 = 60$. What is left is the other part, $100\% - 60\% = 40\%$. *(Gave the empty share)*
- **E** (74.65%): Subtracted the raw decimal from the percentage: $75 - 0.35 = 74.65$. The two figures have to be written in the same notation before they can be combined, and 0.35 of the capacity is 35% of it, so the subtraction is $75\% - 35\% = 40\%$. *(Decimal subtracted without converting)*
- **F** (71.5%): Converted 0.35 to 3.5% by multiplying by 10 rather than 100: $75\% - 3.5\% = 71.5\%$. *(Decimal to percentage slip)*
- **G** (65%): Subtracted from a full battery instead of a three quarters full one: $100\% - 35\% = 65\%$, ignoring the starting charge entirely. *(Starting value ignored)*
- **H** (26.25%): Combined the two figures with a product rather than a difference: $0.75 \times 0.35 = 0.2625$, that is 26.25%. That number is the charge used if the 0.35 were a share of the charge present, so it is an amount removed and not an amount left. *(Multiplied instead of subtracting)*

Common mistake. Subtracting 0.35 from 75 without converting, giving 74.65%, or converting 0.35 to 3.5% and giving 71.5%. A decimal becomes a percentage by multiplying by 100, so 0.35 is 35%, not 3.5% and not 0.35%.

Takeaway. **The Cheat Code:** Put every quantity into one notation before combining them: fraction to percentage multiply by 100, decimal to percentage multiply by 100, and check that both figures are measured against the same whole before you add or subtract.

Time-saving tip. Speed Heuristic: Stay in decimals for the whole item. $\frac{3}{4}$ is 0.75, so the answer is $0.75 - 0.35 = 0.4$, and only the final number needs the single conversion to 40%.

Question 9

M3.11 · Ratio and proportion · ★★★☆☆ · 90 s

A vineyard's yield falls by 20% each year while a disease goes untreated. Three harvests after it appeared the yield is 2560 kg. How large was the last healthy harvest?

- **A.** 6400 kg
- **B.** 4480 kg
- **C.** 3200 kg
- **D.** 4096 kg
- **E.** 5120 kg
- **F.** 4000 kg
- **G.** 6250 kg
- **H.** 5000 kg

Answer H. Key idea

A quantity that falls by a fixed percentage each year is multiplied by the same factor every year, so three years of decay amount to one multiplication by that factor cubed. Undoing the decay therefore means dividing by the cube, not adding the percentage back on. The multiplier for a 20% fall is $1 - 0.20 = 0.8$, and because each year's loss is taken from a smaller amount than the year before, repeated falls never simply add: three falls of 20% remove 48.8%, not 60%.

Worked solution

1. Write down the yearly multiplier

A fall of 20% leaves 80% of the yield standing, so each year multiplies the yield by

$$1 - 0.20 = 0.8$$

2. Compound it over the three harvests

Three successive falls multiply the healthy yield Y by

$$0.8^3 = 0.8 \times 0.8 \times 0.8 = 0.512$$

so $0.512Y = 2560$.

3. Reverse the multiplication

$$Y = \frac{2560}{0.512} = 5000 \text{ kg}$$

Dividing by 0.512 is awkward on paper, so divide by 0.8 three times instead, which is the same as multiplying by 1.25 three times: $2560 \to 3200 \to 4000 \to 5000$.

Sanity check by running the model forwards from 5000 kg: the three harvests come out as 4000 kg, then 3200 kg, then 2560 kg, each one a fifth down on the one before, and the last matches the figure given. The total loss is $1 - 0.512 = 0.488$, that is 48.8%, so the healthy harvest should be a little under twice the present one, and 5000 sits just below $2 \times 2560 = 5120$.

Matches Option **H**.

Why the other options are wrong

- **A** (6400 kg): Three falls of 20% were added into one fall of 60%, giving $2560 \div 0.4 = 6400$. Compounded over three years a 20% annual fall removes 48.8% in total, not 60%. *(Percentages added)*
- **B** (4480 kg): The correct reverse multiplier $\frac{1}{0.8} = 1.25$ was used, but its effect was added as a fixed amount three times: 25% of 2560 is 640, so $2560 + 3 \times 640 = 4480$. Compounding takes 25% of a larger figure each time, so the true steps up are 640, then 800, then 1000, reaching 5000 kg. *(Reverse rise added as a flat amount)*
- **C** (3200 kg): Only one year of decay was reversed: $2560 \div 0.8 = 3200$. That is the yield at the second harvest of the outbreak, not the healthy one, because two further falls still stand between it and the start. *(Only one year reversed)*
- **D** (4096 kg): The fall was treated as a fixed amount rather than a fixed percentage, using 20% of the final yield each time: 20% of 2560 is 512, and $2560 + 3 \times 512 = 4096$. Percentage decay strips a shrinking amount each year, so the earlier losses were larger than 512 kg. *(Fixed amount instead of fixed percentage)*
- **E** (5120 kg): The three year factor $0.8^3 = 0.512$ was rounded to 0.5, so the final yield was simply doubled: $2560 \div 0.5 = 5120$. The 0.012 is worth 120 kg here, and dividing by 0.512 exactly gives 5000 kg. *(Compound factor rounded to a half)*
- **F** (4000 kg): Only two of the three falls were undone: $2560 \div 0.8^2 = 2560 \div 0.64 = 4000$. That is the yield at the first harvest after the disease appeared, and one further division by 0.8 is needed to get back to the healthy one, giving 5000 kg. *(Two years reversed)*
- **G** (6250 kg): Four years of decay were reversed instead of three: $2560 \div 0.8^4 = 2560 \div 0.4096 = 6250$. The disease appeared three harvests before the one quoted, so the factor to undo is $0.8^3 = 0.512$. *(One year too many)*

Common mistake. Treating three years of 20% as a single fall of 60% and dividing by 0.4 to reach 6400 kg. Each year's 20% is taken from a smaller yield than the year before, so the losses cannot be added: their combined effect is $1 - 0.512 = 0.488$, a fall of 48.8%.

Takeaway. **The Cheat Code:** Repeated percentage change is repeated multiplication, so undoing it is repeated division by the same factor. Dividing by 0.8 is the same as multiplying by 1.25, and running the chain forwards at the end costs seconds and catches every slip.

Time-saving tip. Speed Heuristic: never divide by 0.512. Multiply by 1.25 three times instead, which is just adding a quarter each time: $2560 \to 3200 \to 4000 \to 5000$. Every step is a quarter of a round number, so the whole reversal is mental arithmetic.

Question 10

M2.5 · Number · ★★★☆☆ · 90 s

A courier walks 4 blocks east and 2 north. How many routes are possible?

A. 8
B. 360
C. 30
D. 6
E. 64
F. 15
G. 720
H. 10

Answer F. Key idea

A route is nothing more than the order in which the individual block moves are made, so the walk is a list of six moves of which four are east and two are north. Because the four east moves are indistinguishable from one another and so are the two north moves, a route is fixed as soon as you say which positions in the list the north moves occupy. Counting positions rather than listing whole routes turns an awkward enumeration into a short, systematic count.

Worked solution

1. Describe a route as a list of moves

Every route uses exactly 4 east moves and 2 north moves, so it is a list of 6 single block moves. No route can be longer or shorter, since the courier must gain four blocks east and two blocks north in total.

2. Fix a route by the positions of the north moves

The east moves are identical to each other and so are the north moves, so the whole route is determined once you say which two of the six places in the list hold the north moves. Listing those pairs of positions systematically, starting with the first north move in place 1:

$(1,2),(1,3),(1,4),(1,5),(1,6)$, then $(2,3),(2,4),(2,5),(2,6)$, then $(3,4),(3,5),(3,6)$, then $(4,5),(4,6)$, then $(5,6)$.

3. Count the pairs and check

That is $5+4+3+2+1 = 15$ pairs. The same total comes from $\frac{6\times 5}{2} = 15$, where the division by 2 removes the double counting caused by the two north moves being interchangeable.

Sanity check by building the count junction by junction. Along the bottom row of junctions the counts are $1, 1, 1, 1, 1$; the middle row is $1, 2, 3, 4, 5$; the top row is $1, 3, 6, 10, 15$, since each junction is reached from the one to its left plus the one below it. The far corner reads 15, which agrees.

Matches Option **F**.

Why the other options are wrong

- **A** (8): Multiplied the leg lengths, $4 \times 2 = 8$, as though a route were one choice of an east block paired with one choice of a north block. The two legs are not independent single choices. *(Product rule misapplied)*
- **B** (360): All six moves were ordered, $6! = 720$, and the count was then divided by the $2!$ for the two identical north moves but not by the $4!$ for the four identical east moves: $\frac{720}{2} = 360$. Both sets of repeats have to come out, and $\frac{720}{4!\times 2!} = \frac{720}{48} = 15$. *(Only one repeat divided out)*

- **C** (30): Chose the two north positions in order: $6 \times 5 = 30$. Every route is then counted twice, once for each order of the two identical north moves, so this is exactly double the answer. *(Double counting identical moves)*
- **D** (6): Added the two leg lengths, $4 + 2 = 6$. That counts the number of blocks walked, not the number of orders in which they can be walked. *(Counted steps, not routes)*
- **E** (64): Treated each of the 6 moves as a free choice between east and north: $2^6 = 64$. Most of those sequences do not contain exactly four easts and two norths, so they do not end at the right corner. *(Constraint ignored)*
- **G** (720): The six moves were counted as six distinct objects and their orderings totalled: $6! = 720$. Swapping two east moves does not change the route walked, so each route has been counted $4! \times 2! = 48$ times over, and $\frac{720}{48} = 15$. *(Moves treated as distinguishable)*
- **H** (10): The two north moves were placed in different gaps around the four east moves. There are 5 such gaps, one before each east move and one after the last, so this gives $\frac{5 \times 4}{2} = 10$ ways. It discards the 5 routes in which both north moves are taken one straight after the other, both sitting in the same gap, and $10 + 5 = 15$. *(Adjacent north moves excluded)*

Common mistake. Treating the six moves as six distinguishable objects and counting orderings of them, or at the other extreme multiplying the two leg lengths together. The four east moves cannot be told apart, so swapping two of them does not create a new route.

Takeaway. The Cheat Code: A journey made of two kinds of identical steps is counted by choosing which places in the sequence the rarer step occupies: with 6 steps of which 2 are north, that is $\frac{6 \times 5}{2}$, and the junction by junction addition grid gives the same number as a free check.

Time-saving tip. Speed Heuristic: Count the rarer move, not the common one. Two north moves among six places gives $\frac{6 \times 5}{2} = 15$ in one line, whereas working with the four east moves needs $\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2}$ for the same answer.

END OF SOLUTIONS

Questions 11 to 27 are in the full paper

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